# Further Results on Rate 1/N Convolutional Code Constructions With Minimum Required SNR Criterion

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*New good (K, 1/N) convolutional codes for $8 \leqslant K \leqslant 13$ and $2 \leqslant N \leqslant 8$ were found and tabulated which require minimum signal-to-noise ratio (SNR) for given desired bit error rates (BER) with Viterbi decoding. The transfer function bound was used for the BER evaluations.*

## I. Introduction

For a convolutional coding system employing a Viterbi decoder, the decoded bit error rate (BER) is well upper-bounded by the transfer function bound (Refs. 1; 2, Chapter 4)

$$\mathrm{BER} \leqslant c_o \cdot \frac{\partial}{\partial Z} T(D, Z)\bigg|_{D=D_o, Z=1} = c_o \cdot \sum_{i=d_f}^{\infty} a_i \cdot D_o^i \tag{1}$$

where the coefficient $c_o$ and transfer function $T(D,Z)$ depend on the code and type of channel used. $D_o$ is the Bhattacharrya bound (Ref. 2, p. 63) which depends on the channel only, and $d_f$ is the free distance of the code. For an additive white Gaussian noise channel with BPSK signaling (BPSK/AWGN channel) without quantization. $D_o$ and $c_o$ are given by $D_o = \exp(-E_s/N_o)$ and

$$c_o = Q(\sqrt{2d_f \cdot E_s/N_o}) \exp(d_f \cdot E_s/N_o)$$

(Refs. 1; 2, p. 248) where $N_o$ is the one-sided noise power spectral density, $E_s$ is the received signal energy per channel symbol, and

$$Q(w) = \int_w^{\infty} \exp(-t^2/2) \cdot dt/\sqrt{2\pi}$$

Many researchers (e.g., Refs. 3-7) have used the maximum $d_f$ criterion or the criterion of maximum $d_f$ together with minimizing the first few $a_i$'s in Eq. (1), for determinining the goodness of a code in their code search procedures. However, another criterion (Ref. 8) of minimizing required signal-to-noise ratio (SNR) for a given desired BER with the direct use of Eq. (1) for BER evaluation provides much better results. For effective partial code searches in Ref. 8, some known facts were used with a very useful idea that "good codes generate good codes." That is, good rate $1/(N+1)$ codes can be found by extending the code generators of good rate $1/N$ codes. Moreover, the theoretically predicted benefit of coding bandwidth expansion was confirmed with our new codes, whereas the previously reported codes did not uniformly confirm this property.

However, in order to test a code under the above criterion, we have to evaluate the transfer function bound which requires a matrix inversion. Hence, even for a code with short constraint length, considerable amount of computing time has been required. Therefore in Ref. 8 we stopped the code search at constraint length $K=7$. More recently (Ref. 9), a very efficient algorithm for finding the transfer function bound was devised; the algorithm is very fast and requires a much smaller amount of computer memory storage since all the unnecessary operations, such as multiplication-by-zero, etc., are eliminated. This technique enabled us to search for longer constraint length codes.

In the next section, after introducing necessary notations, the partial code searching techniques discussed in Ref. 8 are briefly reviewed. Some additional restrictions which were applied to the searches for longer codes are explained. In the last section, the code search results are summarized in a table and figure where their performance is compared with previously reported codes.

## II. Notations and Partial Code Searching Techniques

A typical nonsystematic, constraint length $K$, rate $1/N$ convolutional encoder, denoted by $(K, 1/N)$ code, is shown in Fig. 1. The code connection box is often represented by an $N \times K$ binary matrix $\mathbf{G}$. Let $G(n)$ and $g(k)$ be the $n$th row and $k$th column vectors of matrix $\mathbf{G}$, and $G(n, k)$ be the element of $n$th row and $k$th column of $\mathbf{G}$ for $n = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, K$. The $n$th bit in the $t$th output vector $y_n^t$ (see Fig. 1), for $t=1,2,\ldots$, is given by:

$$y_n^t = \mathrm{mod}\left\{\sum_{k=1}^{K} G(n, k) \cdot x^{t-k+1}, 2\right\} \tag{2}$$

where $\mathrm{mod}\{u, v\}$ is the remainder when $u$ is divided by $v$, $x^t \in \{0, 1\}$, $t = 1,2,\ldots$ is the encoder input sequence and $x^t = 0$ for $t < 1$ by convention.

The code generator matrix is often represented by $(G(1), G(2), \ldots, G(N))$ with the $G(n)$'s in octal form. We also adopt this notation. For given $K$ and $N$, this code generator $\mathbf{G}$ determines the code performance. By "code search" we imply the search for a $\mathbf{G}$ which provides good performance. The transfer function bounding technique on the BER at the Viterbi decoder output will not be discussed here.

In previous code searches for shorter constraint length codes, we restricted the search space to codes having $g(1) = g(K) = (1, 1, \ldots, 1)$ and eliminated equivalent codes using the obvious facts that "changes in the orders of the $G(n)$'s" or "reversing the order of the $g(k)$'s" gives the same performance. The search space for $(K, 1/N)$ codes was limited by deleting catastrophic codes and codes with too small free distance (smaller than the maximum achieved $d_f$ value of $(K, 1/(N-1))$ codes, or smaller than that of $(K-1, 1/2)$ codes for $N = 2$ cases).

For a given pair of $K$ and $N$, we estimated the values of SNR at which we believed the best code(s) could achieve the prespecified desired values of BER. Search results were listed with the best code at the top. A code was considered to be better than another if the weighted sum of the logarithm of two calculated BER values was smaller. For $N = 2$ cases, the codes in the restricted search space were exhausted. For $N \geqslant 3$ cases, roughly $2^{k-2}$ of the good $(K, 1/(N-1))$ code generators (in the upper portion of the list) were used as seeds to generate the $(K, 1/N)$ code search space. This last restriction was adopted from the observation that "good codes generate good codes."

In the searches for longer constraint length codes, the unquantized BPSK/AWGN channel was assumed as before and the previous procedure was adopted with some minor changes. The first change was in the target values of BER. For shorter codes, the values were $10^{-6}$ and $10^{-3}$. But since the transfer function bound is known to be tight only when the operating SNR is far from the cutoff rate limit, we chose the target BER values to be $10^{-9}$ and $10^{-6}$ for longer codes. Smaller target BER values also reduce the effort in testing a code.

Since, for large $K$, the number of codes in the reduced search space is still too large (it increases exponentially with $K$), we had to further reduce the search space size. In searches for $(K, 1/N)$ codes, codes with $d_f$ smaller than the mid-point between the maximum $d_f$ for $(K, 1/(N-1))$ codes and the upper bound on the $d_f$ for $(K, 1/N)$ codes were deleted. Also the number of seeds was limited to 100 rather than $2^{K-2}$. These limitations increase the possibility of losing better codes, but were required to obtain results in a reasonable length of time.

From the shorter code search results, we made another interesting observation. That is, the number of 1's in the code generator of a good code is equal to, or at most slightly more than, the value of its free distance. This observation was employed for further reducing the search space for longer codes.

## III. Code Search Results and Conclusions

Our code search results are summarized in Table 1 where the code generators and corresponding performances are shown. These are best in the sense of minimizing the required SNR for the upper bound on desired BER of $10^{-9}$ and $10^{-6}$ among the codes searched. Notice that the values of bit SNR ($E_b/N_o$, $E_b = E_S \cdot N$) shown in the table are upper bounds on the required bit SNR for actual target BER values. Also, previously found codes with maximum free distance (Refs. 3-7) are compared to our codes. If more than one code was reported with the same $K$ and $N$, the best one was chosen for comparison. For visual comparisons, the required bit SNR bounds are plotted in Fig. 2 as a function of $N$ (= 1/code rate). The benefit of coding bandwidth expansion becomes more evident with longer constraint length codes.

In conclusion, we have found (1) good codes for values of $N$ and $K$ where no good code had been obtained, and (2) better codes than previously known "best" codes for many of other values of $K$ and $N$. These low-rate codes are expected to have a number of applications, especially for systems having large bandwidth-bit time products such as deep space and spread spectrum communication systems.


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Table 1. Code search results

| (K, 1/N) | Required bit SNR, dB for | | $d_f$ | Notes | Code Generator G in Octal | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1.E-9 | 1.E-6 | | | | | | | | | | |
| (8, 1/2) | 5.987 | 4.481 | 10 | 9,O | 371, | 247 | | | | | | |
| (8, 1/2) | 5.998 | 4.462 | 10 | 6,A | 363, | 255 | | | | | | |
| (8, 1/3) | 5.609 | 4.115 | 16 | 9,6,O | 357, | 251, | 233 | | | | | |
| (8, 1/4) | 5.552 | 4.008 | 22 | 9,A | 353, | 335, | 277, | 231 | | | | |
| (8, 1/4) | 5.553 | 4.005 | 22 | 6,A | 365, | 337, | 271, | 233 | | | | |
| (8, 1/4) | 5.634 | 4.070 | 22 | L | 357, | 313, | 275, | 235 | | | | |
| (8, 1/5) | 5.517 | 3.960 | 28 | 9,A | 367, | 337, | 263, | 251, | 235 | | | |
| (8, 1/5) | 5.522 | 3.956 | 27 | 6,A | 351, | 331, | 265, | 257, | 237 | | | |
| (8, 1/5) | 5.659 | 4.067 | 28 | D | 357, | 323, | 271, | 257, | 233 | | | |
| (8, 1/6) | 5.499 | 3.929 | 33 | 9,A | 363, | 351, | 331, | 265, | 257, | 237 | | |
| (8, 1/6) | 5.520 | 3.926 | 32 | 6,A | 365, | 351, | 337, | 273, | 263, | 221 | | |
| (8, 1/6) | 5.574 | 4.008 | 34 | D | 375, | 357, | 331, | 313, | 253, | 235 | | |
| (8, 1/7) | 5.490 | 3.916 | 39 | 9,A | 373, | 353, | 345, | 335, | 277, | 251, | 231 | |
| (8, 1/7) | 5.513 | 3.906 | 37 | 6,A | 367, | 331, | 311, | 277, | 253, | 235, | 215 | |
| (8, 1/7) | 5.553 | 3.976 | 40 | D | 375, | 357, | 331, | 313, | 275, | 253, | 235 | |
| (8, 1/8) | 5.472 | 3.892 | 44 | 9,A | 371, | 353, | 331, | 323, | 275, | 267, | 237, | 225 |
| (8, 1/8) | 5.473 | 3.889 | 44 | 6,A | 373, | 353, | 335, | 315, | 277, | 251, | 231, | 227 |
| (8, 1/8) | 5.581 | 3.994 | 45 | D | 371, | 357, | 331, | 313, | 275, | 275, | 253, | 235 |
| (9, 1/2) | 5.573 | 4.129 | 12 | 9,O | 753, | 561 | | | | | | |
| (9, 1/2) | 5.603 | 4.122 | 11 | 6,A | 731, | 523 | | | | | | |
| (9, 1/3) | 5.299 | 3.840 | 18 | 9,6,A | 665, | 537, | 471 | | | | | |
| (9, 1/3) | 5.388 | 3.898 | 18 | L | 711, | 663, | 557 | | | | | |
| (9, 1/4) | 5.184 | 3.714 | 24 | 9,A | 765, | 671, | 513, | 473 | | | | |
| (9, 1/4) | 5.219 | 3.709 | 24 | 6,A | 733, | 645, | 571, | 437 | | | | |
| (9, 1/4) | 5.289 | 3.786 | 24 | L | 745, | 733, | 535, | 463 | | | | |
| (9, 1/5) | 5.158 | 3.648 | 29 | 9,A | 751, | 665, | 543, | 537, | 471 | | | |
| (9, 1/5) | 5.160 | 3.647 | 29 | 6,A | 755, | 651, | 637, | 561, | 453 | | | |
| (9, 1/5) | 5.284 | 3.768 | 31 | P | 747, | 675, | 535, | 531, | 467 | | | |
| (9, 1/6) | 5.138 | 3.609 | 35 | 9,6,A | 765, | 721, | 663, | 571, | 513, | 467 | | |
| (9, 1/6) | 5.181 | 3.659 | 37 | P | 727, | 711, | 677, | 553, | 545, | 475 | | |
| (9, 1/7) | 5.126 | 3.590 | 42 | 9,6,A | 763, | 737, | 665, | 551, | 531, | 475, | 427 | |
| (9, 1/7) | 5.229 | 3.671 | 44 | P | 755, | 751, | 737, | 673, | 525, | 463, | 457 | |
| (9, 1/8) | 5.116 | 3.572 | 48 | 9,6,A | 767, | 735, | 665, | 637, | 571, | 551, | 461, | 453 |
| (9, 1/8) | 5.202 | 3.653 | 50 | P | 775, | 717, | 671, | 647, | 625, | 567, | 553, | 513 |
| (10, 1/2) | 5.307 | 3.916 | 12 | 9,A | 1753, | 1151 | | | | | | |
| (10, 1/2) | 5.315 | 3.905 | 12 | 6,J | 1755, | 1363 | | | | | | |
| (10, 1/3) | 5.024 | 3.595 | 19 | 9,A | 1735, | 1261, | 1117 | | | | | |
| (10, 1/3) | 5.025 | 3.589 | 19 | 6,A | 1735, | 1261, | 1163 | | | | | |
| (10, 1/3) | 5.126 | 3.690 | 20 | L | 1633, | 1365, | 1117 | | | | | |
| (10, 1/4) | 4.885 | 3.445 | 26 | 9,A | 1753, | 1547, | 1345, | 1151 | | | | |
| (10, 1/4) | 4.902 | 3.443 | 26 | 6,A | 1753, | 1557, | 1345, | 1151 | | | | |
| (10, 1/4) | 4.985 | 3.549 | 27 | L | 1653, | 1633, | 1365, | 1117, | | | | |
| (10, 1/5) | 4.836 | 3.385 | 33 | 9,A | 1731, | 1537, | 1323, | 1217, | 1135 | | | |
| (10, 1/5) | 4.844 | 3.376 | 32 | 6,A | 1731, | 1621, | 1535, | 1337, | 1123 | | | |
| (10, 1/6) | 4.812 | 3.342 | 40 | 9,6,A | 1755, | 1651, | 1453, | 1371, | 1157, | 1067 | | |
| (10, 1/7) | 4.797 | 3.322 | 46 | 9,A | 1747, | 1731, | 1651, | 1535, | 1337, | 1261, | 1123 | |
| (10, 1/7) | 4.809 | 3.315 | 46 | 6,A | 1713, | 1551, | 1461, | 1365, | 1277, | 1167, | 1075 | |
| (10, 1/8) | 4.785 | 3.294 | 52 | 9,6,A | 1731, | 1621, | 1575, | 1433, | 1327, | 1277, | 1165, | 1123 |

**Table 1. (contd)**

| (K, 1/N) | Required bit SNR, dB for 1.E-9 | 1.E-6 | $d_f$ | Notes | Code Generator G in Octal | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (11, 1/2) | 5.060 | 3.701 | 14 | 9,A | 3643, | 2671 | | | | | | |
| (11, 1/2) | 5.070 | 3.699 | 13 | 6,A | 3723, | 2151 | | | | | | |
| (11, 1/2) | 5.088 | 3.731 | 14 | J | 3645, | 2671 | | | | | | |
| (11, 1/3) | 4.712 | 3.352 | 22 | 9,6,A | 3165, | 2671, | 2373 | | | | | |
| (11, 1/3) | 4.814 | 3.420 | 22 | L | 3175, | 2671, | 2353 | | | | | |
| (11, 1/4) | 4.605 | 3.216 | 28 | 9,A | 3453, | 3077, | 2755, | 2351 | | | | |
| (11, 1/4) | 4.620 | 3.213 | 29 | 6,A | 3453, | 3157, | 2755, | 2351 | | | | |
| (11, 1/4) | 4.722 | 3.293 | 29 | L | 3175, | 2671, | 2353, | 2327 | | | | |
| (11, 1/5) | 4.554 | 3.157 | 36 | 9,A | 3673, | 3161, | 2575, | 2363, | 2265 | | | |
| (11, 1/5) | 4.566 | 3.146 | 35 | 6,A | 3721, | 3165, | 2671, | 2477, | 2153 | | | |
| (11, 1/6) | 4.522 | 3.107 | 42 | 9,A | 3275, | 3165, | 2671, | 2423, | 2277, | 2173 | | |
| (11, 1/6) | 4.537 | 3.099 | 42 | 6,A | 3663, | 3327, | 3161, | 2575, | 2251, | 2177 | | |
| (11, 1/7) | 4.500 | 3.073 | 49 | 9,A | 3625, | 3261, | 3151, | 2733, | 2457, | 2375, | 2167 | |
| (11, 1/7) | 4.505 | 3.068 | 49 | 6,A | 3721, | 3223, | 3165, | 2671, | 2527, | 2277, | 2173 | |
| (11, 1/8) | 4.492 | 3.054 | 57 | 9,6,A | 3651, | 3453, | 3375, | 3167, | 2763, | 2361, | 2265, | 2155 |
| (12, 1/2) | 4.784 | 3.504 | 14 | 9,A | 6765, | 4627 | | | | | | |
| (12, 1/2) | 4.800 | 3.503 | 15 | 6,J | 7173, | 5261 | | | | | | |
| (12, 1/3) | 4.482 | 3.163 | 22 | 9,A | 7473, | 5611, | 4665 | | | | | |
| (12, 1/3) | 4.487 | 3.161 | 23 | 6,A | 6755, | 5271, | 4363 | | | | | |
| (12, 1/3) | 4.512 | 3.180 | 24 | L | 6265, | 5723, | 4767 | | | | | |
| (12, 1/4) | 4.361 | 3.013 | 30 | 9,A | 7635, | 6733, | 5221, | 4627 | | | | |
| (12, 1/4) | 4.363 | 3.011 | 30 | 6,A | 7725, | 6671, | 5723, | 4317 | | | | |
| (12, 1/4) | 4.431 | 3.054 | 32 | L | 7455, | 6265, | 5723, | 4767 | | | | |
| (12, 1/5) | 4.299 | 2.950 | 38 | 9,A | 7725, | 6711, | 5723, | 5513, | 4317 | | | |
| (12, 1/5) | 4.305 | 2.937 | 38 | 6,A | 7725, | 6671, | 5723, | 5321, | 4317 | | | |
| (12, 1/6) | 4.258 | 2.893 | 45 | 9,A | 7725, | 6671, | 5723, | 5161, | 4553, | 4317 | | |
| (12, 1/6) | 4.266 | 2.890 | 45 | 6,A | 7725, | 7341, | 6711, | 5723, | 4533, | 4317 | | |
| (12, 1/7) | 4.235 | 2.859 | 54 | 9,6,A | 7721, | 7325, | 6711, | 5723, | 5337, | 4713, | 4317 | |
| (12, 1/8) | 4.223 | 2.841 | 61 | 9,A | 7721, | 7325, | 6711, | 6545, | 5723, | 5337, | 4713, | 4317 |
| (12, 1/8) | 4.225 | 2.838 | 62 | 6,A | 7725, | 7121, | 6711, | 6277, | 5723, | 5333, | 4735, | 4317 |
| (13, 1/2) | 4.572 | 3.337 | 16 | 9,6,J | 16461, | 12767 | | | | | | |
| (13, 1/3) | 4.251 | 2.993 | 24 | 9,A | 16331, | 12277, | 11565 | | | | | |
| (13, 1/3) | 4.253 | 2.981 | 23 | 6,A | 14331, | 13523, | 10747 | | | | | |
| (13, 1/3) | 4.363 | 3.043 | 24 | L | 17661, | 10675, | 10533 | | | | | |
| (13, 1/4) | 4.119 | 2.841 | 33 | 9,A | 17227, | 14331, | 13277, | 11165 | | | | |
| (13, 1/4) | 4.145 | 2.827 | 32 | 6,A | 16353, | 14751, | 13157, | 10255 | | | | |
| (13, 1/4) | 4.222 | 2.883 | 33 | L | 16727, | 15573, | 12477, | 11145 | | | | |
| (13, 1/5) | 4.055 | 2.753 | 41 | 9,6,A | 17633, | 14471, | 12337, | 11275, | 10565 | | | |
| (13, 1/6) | 4.016 | 2.705 | 49 | 9,6,A | 16365, | 14331, | 13277, | 12467, | 11275, | 10473 | | |
| (13, 1/7) | 3.996 | 2.671 | 58 | 9,6,A | 17661, | 16365, | 14331, | 13277, | 12467, | 11275, | 10473 | |
| (13, 1/8) | 3.979 | 2.652 | 65 | 9,A | 17467, | 16751, | 15345, | 14331, | 13277, | 12475, | 11261, | 10473 |
| (13, 1/8) | 3.986 | 2.650 | 64 | 6,A | 17623, | 16365, | 15221, | 14331, | 13277, | 12467, | 11275, | 10473 |

Notes:

9 Code which minimizes the upperbound on the required SNR for desired BER = 1.E-9 (among searched)
6 Code which minimizes the upperbound on the required SNR for desired BER = 1.E-6 (among searched)
O Found by Odenwalder (Ref. 3)
L Found by Larsen (Ref. 4)
J Found by Johannesson and Paaske (Ref. 5)
D Found by Daut, et. al. (Ref. 6)
P Found by Palazzo (Ref. 7)
A Found by the author

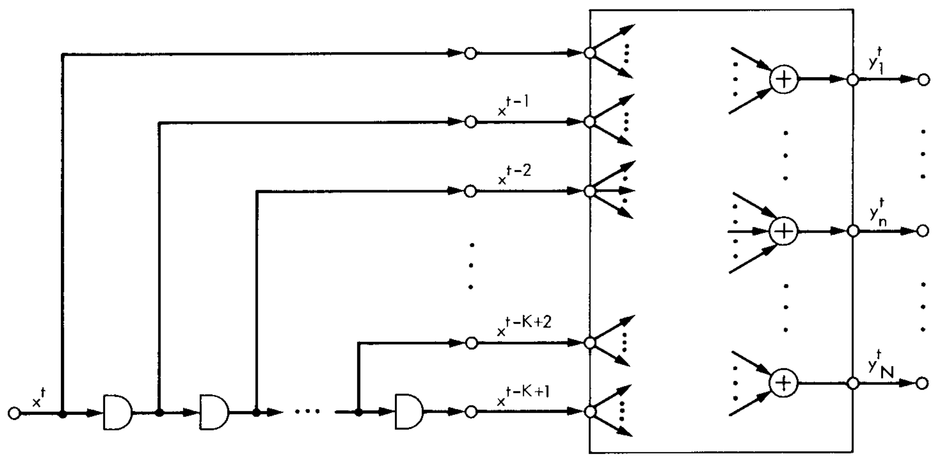


**Fig. 1. A nonsystematic, constraint length *K*, rate 1/*N* convolutional encoder structure**

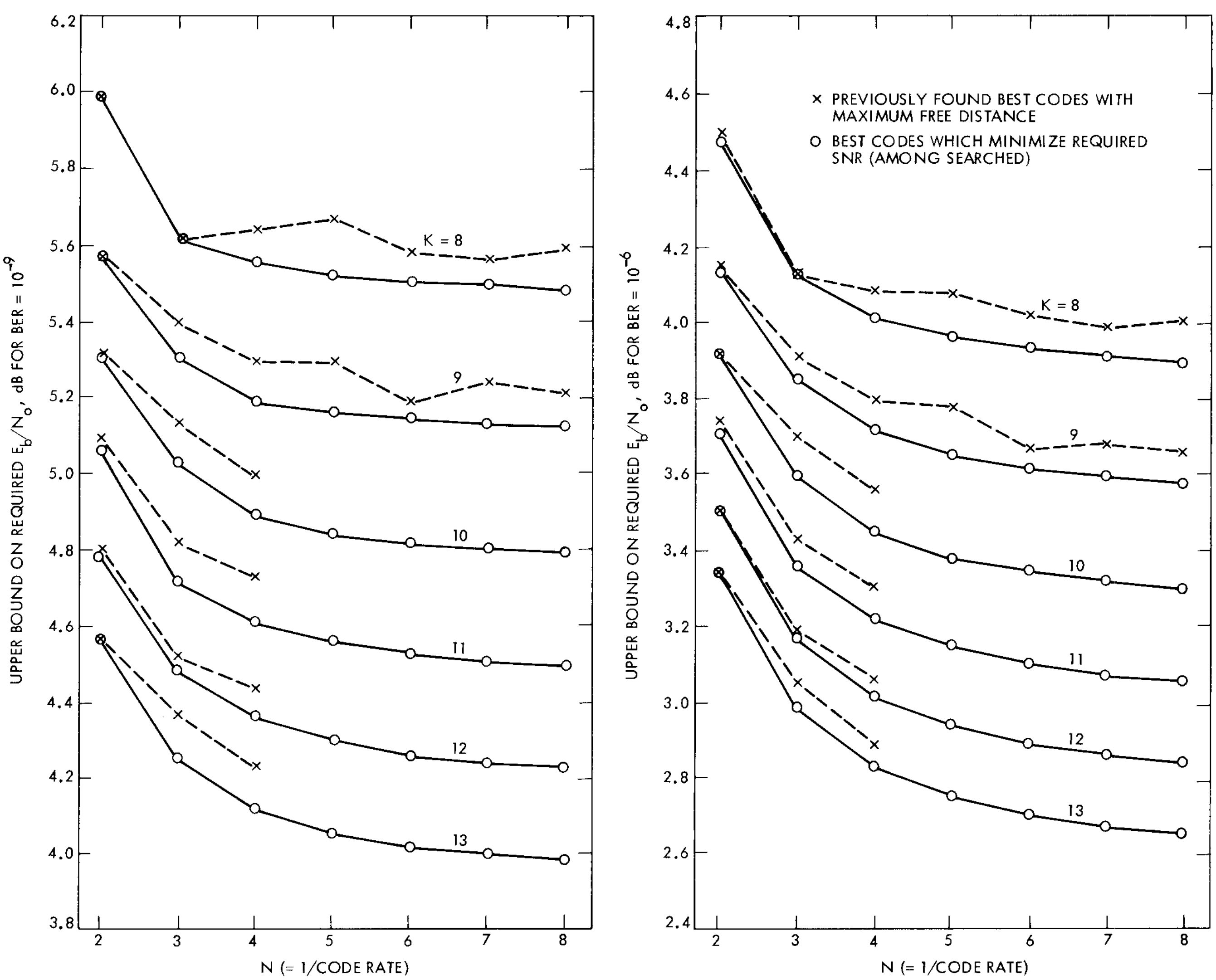


**Fig. 2. Performance comparisons of (*K*, 1/*N*) codes**